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Spherical waves for Dirac–Kähler and Dirac particles, formal relations between boson and fermion solutions†

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Tetrad based equation for Dirac–Kähler particle is solved in spherical coordinates in the flat Minkocski space-time. Spherical solutions of boson type $(J = 0, 1, 2, ...)$ are constructed. After performing a special transformation over spherical boson solutions of the Dirac–Kähler equation, 4×4-matrices $U(x) \Longrightarrow V(x)$, simple linear expansions of the four rows of new representative of the Dirac–Kähler field $V(x)$ in terms of spherical fermion solutions $\Psi_i(x)$ of the four ordinary Dirac equation have been derived. However, this fact cannot be interpreted as the possibility not to distinguish between the Dirac-Kähler field and the system four Dirac fermions. The main formal argument is that the special transformation $(I \otimes S(x))$ involved does not belong to the group of tetrad local gauge transformation for Dirac-Kähler field, 2-rank bispinor under the Lorentz group. Therefore, the linear expansions between boson and fermion functions are not gauge invariant under the group of local tetrad rotations.

1 Spherical solution of the Dirac–Kähler

The Dirac–Kähler field (other terms are Ivanenko–Landau field or vector field of general type) was investigated by many authors – see bibliography in [1, 2]. In the context of the most intriguing question – what does describe this field, a boson or a composite fermion with internal degree of freedom, the goal of the paper is to construct spherical solutions of the Dirac–Kähler field both of boson and fermion type and then to describe relations between them. The problem is solved in the flat Minkowski space. Additionally we specify the case of a curved space time background (3-space of constant positive curvature) where any fermion solutions cannot be constructed.

The Dirac–Kähler, written in a diagonal spherical tetrad of the flat Minkowski space-time, has the form

$$\left[i\gamma^0 \, \partial_t \; + \; i \;\; (\gamma^3 \, \partial_r \; + \; \frac{\gamma^1 J^{31} \; + \; \gamma^2 \; J^{32}}{r} \;) \; + \; \frac{1}{r} \, \Sigma_{\theta,\phi} \; - \; m \right] U(x) = 0 \; , \qquad (1.1a)$$

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$$\Sigma_{\theta,\phi} = \; i\gamma^1 \, \partial_\theta \; + \; \gamma^2 \; \frac{i\partial_\phi \; + \; iJ^{12} \; \cos\theta}{\sin\theta} \; , \qquad J^{12} = (\sigma^{12} \otimes I \; + \; I \otimes \sigma^{12}) \; . \qquad (1.1b)$$

By diagonalizing operators $\mathbf{J}^2, J_3$ of the total angular momentum (first constructing solutions of boson type)

$$J_1 = l_1 \; + \; \frac{iJ^{12} \cos\phi}{\sin\theta} \; , \;\; J_2 = l_2 \; + \; \frac{iJ^{12} \sin\phi}{\sin\theta} \; , \;\; J_3 = l_3 \; , \qquad (1.2a)$$

for the wave function we obtain substitution (details of the relevant general formalism see in [2])

$$U_{\epsilon JM}(t,r,\theta,\phi) = \frac{e^{-i\epsilon t}}{r} \left| \begin{array}{llll} f_{11} \, D_{-1} & f_{12} \, D_0 & f_{13} \, D_{-1} & f_{14} \, D_0 \\ f_{21} \, D_0 & f_{22} \, D_{+1} & f_{23} \, D_0 & f_{24} \, D_{+1} \\ f_{31} \, D_{-1} & f_{32} \, D_0 & f_{33} \, D_{-1} & f_{34} \, D_0 \\ f_{41} \, D_0 & f_{42} \, D_{+1} & f_{43} \, D_0 & f_{44} \, D_{+1} \end{array} \right| , \qquad (1.2b)$$

$f_{ab} = f_{ab}(r)$, $D_\sigma = D^J_{-M,\sigma}(\phi,\theta,0)$, a quantum number J takes in the values $0, \; 1, \; 2, \ldots$ When calculating the action of angular operator, $\Sigma_{\theta,\phi} \; U_{\epsilon JM}$, we need to employ the known formulas [3]

$$\partial_\theta \; D_{-1} = \frac{1}{2}(b \; D_{-2} - a \; D_0) \; , \qquad [(-M + \cos\theta)/\sin\theta] \; D_{-1} = \frac{1}{2}(-b \; D_{-2} - a \; D_0) \; ,$$

$$\partial_\theta \; D_{+1} = \frac{1}{2}(a \; D_0 - b \; D_{+2}) \; , \qquad [(-M - \cos\theta)/\sin\theta] \; D_{+1} = \frac{1}{2}(-a \; D_0 - b \; D_{+2}) \; ,$$

$$\partial_\theta \; D_0 = \frac{1}{2}(a \; D_{-1} - a \; D_{+1}) \; , \qquad [-M/\sin\theta] \; D_0 = \frac{1}{2}(-a \; D_{-1} - a \; D_{+1}) \; ,$$

$$a = \sqrt{J(J+1)} \; , \qquad b = \sqrt{(J-1)(J+1)} \; . \qquad (1.3a)$$

So, for $\Sigma_{\theta,\phi} \; U_{\epsilon JM}$ we get

$$\Sigma_{\theta,\phi} \; U = i\sqrt{J(J+1)} \left| \begin{array}{rrrr} -f_{41} \; D_{-1} & -f_{42} \; D_0 & -f_{43} \; D_{-1} & -f_{44} \; D_0 \\ f_{31} \; D_0 & f_{32} \; D_{+1} & f_{33} \; D_0 & f_{34} \; D_{+1} \\ f_{21} \; D_{-1} & f_{22} \; D_0 & f_{23} \; D_{-1} & f_{24} \; D_0 \\ -f_{11} \; D_0 & -f_{12} \; D_{+1} & -f_{13} \; D_0 & -f_{14} \; D_{+1} \end{array} \right| . \qquad (1.3b)$$

To simplify the problem, on the functions $U_{\epsilon JM}$ let us diagonalyze additionally an operator of P-reflection for the Dirac–Kähler field; in the basis of spherical tetrad it has the form

$$\hat{\Pi}_{sph} = \left| \begin{array}{cccc} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{array} \right| \otimes \left| \begin{array}{cccc} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{array} \right| \otimes \hat{P} \;\; . \qquad (1.4a)$$

From eigenvalue equation $\hat{\Pi}_{sph} U_{\epsilon JM} = \Pi \; U_{\epsilon JM}$, we derive the following restrictions

$$f_{31} = \pm f_{24} \;\; , \;\; f_{32} = \pm f_{23} \;\; , \;\; f_{33} = \pm f_{22} \;\; , \;\; f_{34} = \pm f_{21} \;\; ,$$

$$f_{41} = \pm f_{14} \;\; , \;\; f_{42} = \pm f_{13} \;\; , \;\; f_{43} = \pm f_{12} \;\; , \;\; f_{44} = \pm f_{11} \;\; . \qquad (1.4b)$$

upper sign concerns the value $\Pi = (-1)^{J+1}$, lower concerns the values $\Pi = (-1)^J$. Below, to distinguish between two cases with different parity, we will refer $\Pi = (-1)^{J+1}$ with the symbol $\Delta = -1$, and $\Pi = (-1)^J$ with $\Delta = +1$.

Correspondingly, the substitution will read

$$U_{\epsilon JM\Delta}(t,r,\theta,\phi) = \frac{e^{-i\epsilon t}}{r} \left| \begin{array}{cccc} f_{11}\;D_{-1} & f_{12}\;D_{\;0} & f_{13}\;D_{-1} & f_{14}\;D_{\;0} \\ f_{21}\;D_{\;0} & f_{22}\;D_{+1} & f_{23}\;D_{\;0} & f_{24}\;D_{+1} \\ \Delta f_{24}\;D_{-1} & \Delta f_{23}\;D_{\;0} & \Delta f_{22}\;D_{-1} & \Delta f_{21}\;D_{\;0} \\ \Delta f_{14}\;D_{\;0} & \Delta f_{13}\;D_{+1} & \Delta f_{12}\;D_{\;0} & \Delta f_{11}\;D_{+1} \end{array} \right| . \tag{1.5a}$$

The system of radial equations at $\Delta = +1$ is

$$\epsilon f_{24} - i\frac{d}{dr} f_{24} - i\frac{a}{r} f_{14} - m f_{11} = 0 \; ,$$

$$\epsilon f_{23} - i\frac{d}{dr} f_{23} + i\frac{1}{r} f_{14} - i\frac{a}{r} f_{13} - m f_{12} = 0 \; ,$$

$$\epsilon f_{22} - i\frac{d}{dr} f_{22} - i\frac{a}{r} f_{12} - m f_{13} = 0 \; ,$$

$$\epsilon f_{21} - i\frac{d}{dr} f_{21} + i\frac{1}{r} f_{12} - i\frac{a}{r} f_{11} - m f_{14} = 0 \; ,$$

$$\epsilon f_{14} + i\frac{d}{dr} f_{14} + i\frac{1}{r} f_{23} + i\frac{a}{r} f_{24} - m f_{21} = 0 \; ,$$

$$\epsilon f_{13} + i\frac{d}{dr} f_{13} + i\frac{a}{r} f_{23} - m f_{22} = 0 \; ,$$

$$\epsilon f_{12} + i\frac{d}{dr} f_{12} + i\frac{1}{r} f_{21} + i\frac{a}{r} f_{22} - m f_{23} = 0 \; ,$$

$$\epsilon f_{11} + i\frac{d}{dr} f_{11} + i\frac{1}{r} f_{21} + i\frac{a}{r} f_{21} - m f_{24} = 0 \; . \tag{1.5b}$$

Having changed m into $-m$, we produce analogous equations at $\Delta = -1$.

Now let us translate equations to the following new combinations

$$\begin{aligned} A &= (f_{11} + f_{22})/\sqrt{2} \; , \qquad & B &= (f_{11} - f_{22})/i\sqrt{2} \; , \\ C &= (f_{12} + f_{21})/\sqrt{2} \; , \qquad & D &= (f_{11} - f_{22})/i\sqrt{2} \; , \\ K &= (f_{13} + f_{24})/\sqrt{2} \; , \qquad & L &= (f_{13} - f_{24})/i\sqrt{2} \; , \\ M &= (f_{14} + f_{23})/\sqrt{2} \; , \qquad & N &= (f_{14} + f_{23})/i\sqrt{2} \; . \end{aligned} \tag{1.6a}$$

As result, we obtain equations without imaginary i

$$\epsilon K - \frac{dL}{dr} + \frac{a}{r} N - mA = 0 \; ,$$

$$\epsilon L + \frac{dK}{dr} + \frac{a}{r} N + mB = 0 \; ,$$

$$\epsilon A - \frac{dB}{dr} + \frac{a}{r} D - mK = 0 \; ,$$

$$\epsilon B + \frac{dA}{dr} + \frac{a}{r} C + mL = 0 \; ,$$

$$\epsilon M - \frac{dN}{dr} + \frac{1}{r} N + \frac{a}{r} L - mC = 0 \; ,$$

$$\epsilon N + \frac{dM}{dr} + \frac{1}{r} M + \frac{a}{r} K + mD = 0 \; ,$$

$$\epsilon C - \frac{dD}{dr} + \frac{1}{r} D + \frac{a}{r} B - mM = 0 \; ,$$

$$\epsilon D + \frac{dC}{dr} + \frac{1}{r} C + \frac{a}{r} A + mN = 0 \; . \qquad 1.6b$$

Note. that eqs. (1.6b) permit the following linear constraints

$$A = \lambda\, K \; , \;\; B = \lambda\, L \; , \;\; C = \lambda\, M \; , \;\; D = \lambda\, N \; , \qquad (1.7a)$$

where $\lambda = \pm 1$. In particular, at $\lambda = +1$ we get a system of four equations

$$\frac{dK}{dr} + \frac{a}{r} M + (\epsilon + m) L = 0 \; ,$$

$$\frac{dL}{dr} - \frac{a}{r} N - (\epsilon - m) K = 0 \; ,$$

$$(\frac{d}{dr} + \frac{1}{r}) M + \frac{a}{r} K + (\epsilon + m) N = 0 \; ,$$

$$(\frac{d}{dr} - \frac{1}{r}) N - \frac{a}{r} L - (\epsilon - m) M = 0 \; . \qquad (1.7b)$$

By formal changing m into $-m$, we will obtain equations for the case $\lambda = -1$.

Taking these restrictions into account, the substitution for solution $U^{\lambda}_{\epsilon JM\Delta}$ can be written in a simpler form

$$U^{\lambda}_{\epsilon JM\Delta}(t, r, \theta, \phi) = \frac{e^{-i\epsilon t}}{r\sqrt{2}} \times$$

$$\left| \begin{array}{cccc} \lambda\, (K + iL)\, D_{-1} & \lambda\, (M + iN)\, D_{\;0} & (K + iL)\, D_{-1} & (M + iN)\, D_{\;0} \\ \lambda\, (M - iN)\, D_{\;0} & \lambda\, (K - iL)\, D_{+1} & (M - iN)\, D_{\;0} & (K - iL)\, D_{+1} \\ \Delta\, (K - iL)\, D_{-1} & \Delta\, (M - iN)\, D_{\;0} & \Delta\, \lambda\, (K - iL)\, D_{-1} & \Delta\, \lambda\, (M - iN)\, D_{\;0} \\ \Delta\, (M + iN)\, D_{\;0} & \Delta\, (K + iL)\, D_{+1} & \Delta\, \lambda\, (M + iN)\, D_{\;0} & \Delta\, \lambda\, (K + iL)\, D_{+1} \end{array} \right| . \qquad (1.7c)$$

Equations (1.7b) can be solved with the use of two different substitutions

$$I. \;\; \sqrt{J+1}\; K(r) = f(r) \;\; , \;\; \sqrt{J+1}\;\; L(r) = g(r) \;\; ,$$
$$\sqrt{J}\; M(r) = f(r) \;\; , \;\; \sqrt{J}\; N(r) = g(r) \; ; \qquad (1.8a)$$

$$II. \;\; \sqrt{J}\; K(r) = f(r) \;\; , \;\; \sqrt{J}\; L(r) = g(r) \;\; ,$$
$$\sqrt{J+1}\; M(r) = -f(r) \;\; , \;\; \sqrt{J+1}\; N(r) = -g(r) \; . \qquad (1.8b)$$

In the case (1.8a), we get

$$I. \qquad (\frac{d}{dr} + \frac{J+1}{r}) f + (\epsilon + m) g = 0 \; ,$$

$$(\frac{d}{dr} - \frac{J+1}{r}) g - (\epsilon - m) f = 0 \; . \tag{1.9a}$$

and similarly for (1.8b) we obtain

$$II. \qquad (\frac{d}{dr} - \frac{J}{r}) f + (\epsilon + m) g = 0 \; ,$$

$$(\frac{d}{dr} + \frac{J}{r}) g - (\epsilon - m) f = 0 \; . \tag{1.9b}$$

Remember that eqs. (1.9a,b) refer to the case $\Delta = +1 \quad \lambda = +1$.

Thus, at fixed quantum numbers $(\epsilon,\ J,\ M,\ \Delta)$, there exist four types of solutions: due to two number for $\lambda = \pm 1$ and due to existence of two substitutions I and II (see (1.8a,b)).

Solutions of the type I are described by

$$U^{I.\lambda}_{\epsilon JM\Delta}(x) = \frac{e^{i\epsilon t}}{r} \times$$

$$\left| \begin{array}{cccc} \lambda\, D_{-1}/\sqrt{J+1} & \lambda\, D_{\;0}/\sqrt{J} & D_{-1}/\sqrt{J+1} & D_{\;0}/\sqrt{J} \\ \lambda\, D_{\;0}/\sqrt{J} & \lambda\, D_{+1}/\sqrt{J+1} & D_{\;0}/\sqrt{J} & D_{+1}/\sqrt{J+1} \\ D_{-1}/\sqrt{J+1} & D_{\;0}/\sqrt{J} & \lambda\, D_{-1}/\sqrt{J+1} & \lambda\, D_{\;0}/\sqrt{J} \\ D_{\;0}/\sqrt{J} & D_{+1}/\sqrt{J+1} & \lambda\, D_{\;0}/\sqrt{J} & \lambda\, D_{+1}/\sqrt{J+1} \end{array} \right| \begin{array}{l} \leftarrow \quad (f+ig)/\sqrt{2} \\ \leftarrow \quad (f-ig)/\sqrt{2} \\ \leftarrow \Delta(f-ig)/\sqrt{2} \\ \leftarrow \Delta(f+ig)/\sqrt{2} \end{array} \; ; \tag{1.10a}$$

where all elements of each line should by multiplied by a function from the left; at $\Delta = +1, \lambda = +1$, the functions f and g obey eqs. (1.9a), whereas for three remaining cases we should perform in (1.9a) formal changes in accordance with the rules

$$(\; \Delta = -1, \; \lambda = +1 \;) \qquad m \; \rightarrow \; -m \; ;$$

$$(\; \Delta = +1, \; \lambda = -1 \;) \qquad m \; \rightarrow -m \; ;$$

$$(\; \Delta = -1, \; \lambda = -1 \;) \qquad m \; \rightarrow +m \; . \tag{1.10b}$$

Analogously, for solutions of the second type we have

$$U^{II.\lambda}_{\epsilon JM\Delta}(x) = \frac{e^{-i\epsilon t}}{r} \times$$

$$\left| \begin{array}{cccc} -\lambda\, D_{-1}/\sqrt{J} & \lambda\, D_{\;0}/\sqrt{J+1} & D_{-1}/\sqrt{J} & -D_{\;0}/\sqrt{J+1} \\ \lambda\, D_{\;0}/\sqrt{J+1} & -\lambda\, D_{+1}/\sqrt{J} & -D_{\;0}/\sqrt{J+1} & D_{+1}/\sqrt{J} \\ D_{-1}/\sqrt{J} & -D_{\;0}/\sqrt{J+1} & -\lambda\, D_{-1}/\sqrt{J} & \lambda\, D_{\;0}/\sqrt{J+1} \\ -D_{\;0}/\sqrt{J+1} & D_{+1}/\sqrt{J} & \lambda\, D_{\;0}/\sqrt{J+1} & -\lambda\, D_{+1}/\sqrt{J} \end{array} \right| \begin{array}{l} \leftarrow \quad (f+ig)/\sqrt{2} \\ \leftarrow \quad (f-ig)/\sqrt{2} \\ \leftarrow \Delta(f-ig)/\sqrt{2} \\ \leftarrow \Delta(f+ig)/\sqrt{2} \end{array} \; . \tag{1.10c}$$

At $\Delta = +1,\ \lambda = +1$, the functions f and g obey eqs. (1.9b); in three remaining cases one should use the rules (1.10b).

The case of minimal value $J = 0$ needs a separate consideration. Indeed, initial substitution for the wave function $U_{\epsilon 00}(x)$ turns to be independent on angular variables

$$U_{\epsilon 00}(t,r) = \frac{e^{-i\epsilon t}}{r} \left| \begin{array}{cccc} 0 & f_{12} & 0 & f_{14} \\ f_{21} & 0 & f_{23} & 0 \\ 0 & f_{32} & 0 & f_{34} \\ f_{41} & 0 & f_{43} & 0 \end{array} \right| . \tag{1.11}$$

The operator of spacial inversion, being only a matrix operation, permits to separate functions (1.11) in two classes – eigenvalue equation $\hat{\Pi}\, U_{\epsilon 00} = \Pi\, U_{\epsilon 00}$ gives:

$\Pi = +1$ $(\Delta = +1)$,

$$f_{32} = +f_{23}\ , \quad f_{34} = +f_{21}\ , \quad f_{41} = +f_{14}\ , \quad f_{43} = +f_{12}\ \ ; \tag{1.12a}$$

$\Pi = -1 (\Delta = -1)$,

$$f_{32} = -f_{23}\ , \quad f_{34} = -f_{21}\ , \quad f_{41} = -f_{14}\ , \quad f_{43} = -f_{12}\ \ . \tag{1.12b}$$

Allowing for relation $\Sigma_{\theta,\phi}\, U_{\epsilon 00} = 0$, we derive the radial system (for states with $J09$, functions $A,\ B,\ K,\ L$ in (1.6a)) vanish identically)

$$\epsilon\, M - \frac{dN}{dr} + \frac{N}{r} - m\, C = 0\ \ ,$$

$$\epsilon\, N + \frac{dM}{dr} + \frac{M}{r} + m\, D = 0\ ,$$

$$\epsilon\, C - \frac{dD}{dr} + \frac{D}{r} - m\, M = 0\ ,$$

$$\epsilon\, D + \frac{dC}{dr} + \frac{C}{r} - m\, N = 0\ . \tag{1.12c}$$

To obtain equations when $\Delta = -1$, one should change m into $-m$.

The system (1.12) can be simplified by two substitutions:

$C = +M\ ,\ D = +N$ $(\lambda = +1)$

$$(\frac{d}{dr} + \frac{1}{r})M + (\epsilon + m)N = 0\ ,$$

$$(\frac{d}{dr} - \frac{1}{r})N - (\epsilon - m)M = 0\ ; \tag{1.13a}$$

$C = -M,\ D = -N$ $(\lambda = -1)$

$$(\frac{d}{dr} + \frac{1}{r})M + (\epsilon - m)N = 0\ ,$$

$$(\frac{d}{dr} - \frac{1}{r})N - (\epsilon + m)M = 0\ . \tag{1.13b}$$

Thus, at $J = 0$ and a fixed parity, there exist two different solutions (doubling by $\lambda = \pm 1$):

$$U^{\lambda}_{\epsilon 00\Delta}(t,r) = \frac{e^{-i\epsilon t}}{r} \left| \begin{array}{cccc} 0 & \lambda & 0 & 1 \\ \lambda & 0 & 1 & 0 \\ 0 & 1 & 0 & \lambda \\ 1 & 0 & \lambda & 0 \end{array} \right| \begin{array}{l} \leftarrow \ \ (M + iN)/\sqrt{2} \\ \leftarrow \ \ (M - iN)/\sqrt{2} \\ \leftarrow \Delta(M - iN)/\sqrt{2} \\ \leftarrow \Delta(M + iN)/\sqrt{2} \end{array} . \tag{1.14}$$

2 On relations between boson and fermion solutions

Now let us relate the above spherical solutions of boson type with spherical solutions of the ordinary Dirac equation [2]

$$
\Psi_{\epsilon jm\delta}(x) = \frac{e^{-i\epsilon t}}{r} \left| \begin{array}{c} D_{-1/2} \\ D_{+1/2} \\ D_{-1/2} \\ D_{+1/2} \end{array} \right| \begin{array}{l} \leftarrow \; (F+iG)/\sqrt{2} \\ \leftarrow \; (F-iG)/\sqrt{2} \\ \leftarrow \delta(F-iG)/\sqrt{2} \\ \leftarrow \delta(F+iG)/\sqrt{2} \end{array} \, , \qquad (2.1a)
$$

where $\delta = +1$ refers to the parity $P = (-1)^{j+1}$, and $\delta = -1$ refers to the parity $P = (-1)^{j}$. Radial equations for F and G at $\delta = +1$ are

$$
\left(\frac{d}{dr} + \frac{j+1/2}{r} \right) F + (\epsilon + m)\, G = 0 \, ,
$$

$$
\left(\frac{d}{dr} - \frac{j+1/2}{r} \right) G - (\epsilon - m)\, F = 0 \, ; \qquad (2.1b)
$$

changing the sign at m in (2.1b) we obtain equation for states with different parity (the case $\delta = -1$).

In order to connect explicitly the above boson solutions of the Dirac–Kähler field with spherical solutions of the (four) Dirac equations, one must perform over the matrix $U(x)$ a special transformation $U(x) \;\; \rightarrow \;\; V(x)$, choosing it so that in a new representation the Dirac–Kähler equation is splitted into four separated Dirac-Like equation; then there arises possibility to decompose four rows of the 4×4-matrix $V(x)$, related with the Dirac-Kähler equation, in terms of solutions of four Dirac equation.

The transformation we need has the form

$$
V(x) = (I \otimes S(x))\, U(x) \, , \qquad S(x) = \left| \begin{array}{cc} B(x) & 0 \\ 0 & B(x) \end{array} \right| \, ,
$$

$$
B(x) = \left| \begin{array}{cc} \cos\frac{\theta}{2}\, e^{-i\phi/2} & \sin\frac{\theta}{2}\, e^{-i\phi/2} \\ -\sin\frac{\theta}{2}\, e^{+i\phi/2} & \cos\frac{\theta}{2}\, e^{+i\phi/2} \end{array} \right| \, . \qquad (2.2a)
$$

Spherical bispinor connection Γ_α

$$
\Gamma_t = 0 \, , \;\; \Gamma_r = 0 \, , \;\; \Gamma_\theta = \sigma^{12} \, , \;\; \Gamma_\phi = \sin\theta \, \sigma^{32} + cos\theta \, \sigma^{12}
$$

entering the Dirac–Kähler equation translated to $V(x)$-representation

$$
\{ \, [\, i\gamma^{\alpha}(x) \, (\partial_\alpha \; + \; \Gamma_\alpha(x)) \; - \; m \,] \, V(x)
$$

$$
+ i \, \gamma^{\alpha}(x) \, V(x) \, [\, S(x) \, \Gamma_\alpha(x) \, S^{-1}(x) \; + \; S(x) \, \partial_\alpha \, S^{-1}(x) \,] \, \} = 0 \qquad (2.2b)
$$

will make to zero the following term

$$
S(x) \, \Gamma_\alpha(x) \, S^{-1}(x) \; + \; S(x) \, \partial_\alpha \, S^{-1}(x) = 0
$$

and we obtain what needed

$$[\ i\gamma^{\alpha}(x)\ (\partial_{\alpha}\ +\ \Gamma_{\alpha}(x))\ -\ m\ ]\ V(x) = 0\ . \tag{2.2c}$$

Thus, the task consists in the following:

1) first, we should translate the above spherical solutions in U-form to corresponding V-form;
2) second, we should expand four rows of the matrix V in terms of Dirac spherical waves.

With the use of (2.2a), the matrix $U_{\epsilon JM}$ in (1.3b) will assumes the forn (we have written V_{ij} by rows)

$$(V_{i1}) = \left| \begin{array}{c} f_{11} D_{-1} \cos \frac{\theta}{2} e^{-i\phi/2} \ - \ f_{12} D_0 \sin \frac{\theta}{2} e^{-i\phi/2} \\ f_{21} D_0 \cos \frac{\theta}{2} e^{-i\phi/2} \ - \ f_{22} D_{+1} \sin \frac{\theta}{2} e^{-i\phi/2} \\ f_{31} D_{-1} \cos \frac{\theta}{2} e^{-i\phi/2} \ - \ f_{32} D_0 \sin \frac{\theta}{2} e^{-i\phi/2} \\ f_{41} D_0 \cos \frac{\theta}{2} e^{-i\phi/2} \ - \ f_{42} D_{+1} \sin \frac{\theta}{2} e^{-i\phi/2} \end{array} \right| ,$$

$$(V_{i2}) = \left| \begin{array}{c} f_{11} D_{-1} \sin \frac{\theta}{2} e^{+i\phi/2} \ + \ f_{12} D_0 \cos \frac{\theta}{2} e^{+i\phi/2} \\ f_{21} D_0 \sin \frac{\theta}{2} e^{+i\phi/2} \ + \ f_{22} D_{+1} \cos \frac{\theta}{2} e^{+i\phi/2} \\ f_{31} D_{-1} \sin \frac{\theta}{2} e^{+i\phi/2} \ + \ f_{32} D_0 \cos \frac{\theta}{2} e^{+i\phi/2} \\ f_{41} D_0 \sin \frac{\theta}{2} e^{+i\phi/2} \ + \ f_{42} D_{+1} \cos \frac{\theta}{2} e^{+i\phi/2} \end{array} \right| ,$$

$$(V_{i3}) = \left| \begin{array}{c} f_{13} D_{-1} \cos \frac{\theta}{2} e^{-i\phi/2} \ - \ f_{14} D_0 \sin \frac{\theta}{2} e^{-i\phi/2} \\ f_{23} D_0 \cos \frac{\theta}{2} e^{-i\phi/2} \ - f_{24} D_{+1} \sin \frac{\theta}{2} e^{-i\phi/2} \\ f_{33} D_{-1} \cos \frac{\theta}{2} e^{-i\phi/2} \ - \ f_{34} D_0 \sin \frac{\theta}{2} e^{-i\phi/2} \\ f_{43} D_0 \cos \frac{\theta}{2} e^{-i\phi/2} \ - f_{44} D_{+1} \sin \frac{\theta}{2} e^{-i\phi/2} \end{array} \right| ,$$

$$(V_{i4}) = \left| \begin{array}{c} f_{13} D_{-1} \sin \frac{\theta}{2} e^{+i\phi/2} \ + \ f_{14} D_0 \cos \frac{\theta}{2} e^{+i\phi/2} \\ f_{23} D_0 \sin \frac{\theta}{2} e^{+i\phi/2} \ + \ f_{24} D_{+1} \cos \frac{\theta}{2} e^{+i\phi/2} \\ f_{33} D_{-1} \sin \frac{\theta}{2} e^{+i\phi/2} \ + \ f_{34} D_0 \cos \frac{\theta}{2} e^{+i\phi/2} \\ f_{43} D_0 \sin \frac{\theta}{2} e^{+i\phi/2} \ + \ f_{44} D_{+1} \cos \frac{\theta}{2} e^{+i\phi/2} \end{array} \right| .$$

We will apply 8 formulas relating D-functions of integer and half-integer j [**?**]; two of them are written down below

$$\cos \frac{\theta}{2} e^{i\phi/2}\ D^{J}_{-M,0} = \sqrt{\frac{J(J-M)}{2J+1}}\ D^{J-1/2}_{-M-1/2,-1/2}\ +\ \sqrt{\frac{(J+1)(J+M+1)}{2J+1}}\ D^{J+1/2}_{-M-1/2,-1/2}\ ,$$

$$\cos \frac{\theta}{2} e^{i\phi/2}\ D^{J}_{-M,+1} = \sqrt{\frac{(J+1)(J-M)}{2J+1}}\ D^{J-1/2}_{-M-1/2,+1/2}\ +\ \sqrt{\frac{J(J+M+1)}{2J+1}}\ D^{J+1/2}_{-M-1/2,+1/2}\ .$$

With the help of those relations, for (V_{ij}) we obtain (the factor $e^{-i\epsilon t}/r$ is omitted)

$$V_{\epsilon jm} = V_{\epsilon JM}^{(J-1/2)} + V_{\epsilon JM}^{(J+1/2)} , \tag{2.3}$$

where

$$(V_{i1}^{(J-1/2)}) = \sqrt{\frac{J+M}{2J+1}} \left| \begin{array}{c} (\sqrt{J+1}f_{11} - \sqrt{J}f_{12})\, D^{J-1/2}_{-M+1/2,-1/2} \\ (\sqrt{J}f_{21} - \sqrt{J+1}f_{22})\, D^{J-1/2}_{-M+1/2,+1/2} \\ (\sqrt{J+1}f_{31} - \sqrt{J}f_{32})\, D^{J-1/2}_{-M+1/2,-1/2} \\ (\sqrt{J}f_{41} - \sqrt{J+1}f_{42})\, D^{J-1/2}_{-M+1/2,+1/2} \end{array} \right| ,$$

$$(V_{i2}^{(J-1/2)}) = \sqrt{\frac{J-M}{2J+1}} \left| \begin{array}{c} -(\sqrt{J+1}f_{11} - \sqrt{J}f_{12})\, D^{J-1/2}_{-M-1/2,-1/2} \\ -(\sqrt{J}f_{11} - \sqrt{J+1}f_{22})\, D^{J-1/2}_{-M-1/2,+1/2} \\ -(\sqrt{J+1}f_{21} - \sqrt{J}f_{22})\, D^{J-1/2}_{-M-1/2,-1/2} \\ -(\sqrt{J}f_{41} - \sqrt{J+1}f_{42})\, D^{J-1/2}_{-M-1/2,+1/2} \end{array} \right| ,$$

$$(V_{i3}^{(J-1/2)}) = \sqrt{\frac{J+M}{2J+1}} \left| \begin{array}{c} (\sqrt{J+1}f_{13} - \sqrt{J}f_{14})\, D^{J-1/2}_{-M+1/2,-1/2} \\ (\sqrt{J}f_{23} - \sqrt{J+1}f_{24})\, D^{J-1/2}_{-M+1/2,+1/2} \\ (\sqrt{J+1}f_{33} - \sqrt{J}f_{34})\, D^{J-1/2}_{-M+1/2,-1/2} \\ (\sqrt{J}f_{43} - \sqrt{J+1}f_{44})\, D^{J-1/2}_{-M+1/2,+1/2} \end{array} \right| ,$$

$$(V_{i4}^{(J-1/2)}) = \sqrt{\frac{J-M)}{2J+1}} \left| \begin{array}{c} -(\sqrt{J+1}f_{13} - \sqrt{J}f_{14}) D^{J-1/2}_{-M-1/2,-1/2} \\ -(\sqrt{J}f_{23} - \sqrt{J+1}f_{24}) D^{J-1/2}_{-M-1/2,+1/2} \\ -(\sqrt{J+1}f_{33} - \sqrt{J}f_{34}) D^{J-1/2}_{-M-1/2,-1/2} \\ -(\sqrt{J}f_{44} - \sqrt{J+1}f_{44}) D^{J-1/2}_{-M-1/2,+1/2} \end{array} \right|$$

and

$$(V_{i2}^{(J+1/2)}) = \sqrt{\frac{J-M+1}{2J+1}} \left| \begin{array}{c} (\sqrt{J}f_{11} + \sqrt{J+1}f_{12})\, D^{J+1/2}_{-M+1/2,-1/2} \\ (\sqrt{J+1}f_{21} + \sqrt{J}f_{22})\, D^{J+1/2}_{-M+1/2,+1/2} \\ (\sqrt{J}f_{31} + \sqrt{J+1}f_{32})\, D^{J+1/2}_{-M+1/2,-1/2} \\ (\sqrt{J+1}f_{41} + \sqrt{J}f_{42})\, D^{J+1/2}_{-M+1/2,+1/2} \end{array} \right| ,$$

$$(V_{i2}^{(J+1/2)}) = \sqrt{\frac{J+M+1}{2J+1}} \left| \begin{array}{c} (\sqrt{J}f_{11} + \sqrt{J+1}f_{12})\, D^{J+1/2}_{-M-1/2,-1/2} \\ (\sqrt{J+1}f_{11} + \sqrt{J}f_{22})\, D^{J+1/2}_{-M-1/2,+1/2} \\ (\sqrt{J}f_{21} + \sqrt{J+1}f_{22})\, D^{J+1/2}_{-M-1/2,-1/2} \\ (\sqrt{J+1}f_{41} + \sqrt{J}f_{42})\, D^{J+1/2}_{-M-1/2,+1/2} \end{array} \right| ,$$

$$(V_{i3}^{(J+1/2)}) = \sqrt{\frac{J-M+1}{2J+1}} \left| \begin{array}{c} (\sqrt{J}f_{13} + \sqrt{J+1}f_{14})\, D^{J+1/2}_{-M+1/2,-1/2} \\ (\sqrt{J+1}f_{23} + \sqrt{J}f_{24})\, D^{J+1/2}_{-M+1/2,+1/2} \\ (\sqrt{J}f_{33} + \sqrt{J+1}f_{34})\, D^{J+1/2}_{-M+1/2,-1/2} \\ (\sqrt{J+1}f_{43} + \sqrt{J}f_{44})\, D^{J+1/2}_{-M+1/2,+1/2} \end{array} \right| ,$$

$$
(V_{i4}^{(J+1/2)}) = \sqrt{\frac{J+M+1}{2J+1}} \left| \begin{array}{l} (\sqrt{J}f_{13} \ + \ \sqrt{J+1}f_{14}) \, D^{J+1/2}_{-M-1/2,-1/2} \\ (\sqrt{J+1}f_{23} \ + \ \sqrt{J}f_{24}) \, D^{J+1/2}_{-M-1/2,+1/2} \\ (\sqrt{J}f_{33} \ + \ \sqrt{J+1}f_{34}) \, D^{J+1/2}_{-M-1/2,-1/2} \\ (\sqrt{J+1}f_{43} \ + \ \sqrt{J}f_{44}) \, D^{J+1/2}_{-M-1/2,+1/2} \end{array} \right| .
$$

Now the functions f_{ab} in (1.3) should be taken in accordance with the substitutions given below

$$
f_{ab} = \left| \begin{array}{rrrr} \lambda\,(K+iL) & \lambda\,(M+iN) & (K+iL) & (M+iN) \\ \lambda\,(M-iN) & \lambda\,(K-iL) & (M-iN) & (K-iL) \\ \Delta\,(K-iL) & \Delta\,(M-iN) & \Delta\lambda\,(K-iL) & \Delta\lambda(M-iN) \\ \Delta\,(M+iN) & \Delta\,(K+iL) & \Delta\lambda(M+iN) & \Delta\lambda(K+iL) \end{array} \right| ,
$$

where $\lambda = \pm 1$ and $\Delta = \pm 1$. Thus, from (1.3) it follows

$$
V^{(J-1/2)}_{\epsilon JM\Delta\lambda} = \left| \begin{array}{rrrr} \lambda\,\Omega & -\lambda\,\Xi & \Omega & \Xi \\ -\lambda\,\Upsilon & \lambda\,Z & \Upsilon & Z \\ \Omega & \Xi & \lambda\,\Omega & -\lambda\,\Xi \\ \Upsilon & Z & -\lambda\,\Upsilon & \lambda\,Z \end{array} \right| \begin{array}{l} \leftarrow \ H^{+} \\ \leftarrow \ H^{-} \\ \leftarrow \Delta H^{+} \\ \leftarrow \Delta H^{-} \end{array} , \qquad (2.4a)
$$

where symbols Ω, Ξ, Υ, Z stand for expressions

$$
\Omega = \sqrt{\frac{J+M}{J(J+1)}}\, D^{J-1/2}_{-M+1/2,-1/2} \ , \ \ Y = \sqrt{\frac{J+M}{J(J+1)}}\, D^{J-1/2}_{-M+1/2,+1/2} \ ,
$$

$$
\Xi = \sqrt{\frac{J-M}{J(J+1)}}\, D^{J-1/2}_{-M-1/2,-1/2} \ , \ \ Z = \sqrt{\frac{J-M}{J(J+1)}}\, D^{J-1/2}_{-M-1/2,-1/2} \ , \qquad (2.4b)
$$

and $H^{\pm}(r)$ represent

$$
H^{\pm}(r) = \sqrt{\frac{J(J+1)}{2J+1}} \ \Big[\ \sqrt{J+1}\,(K \pm iL) \ - \ \sqrt{J}\,(M \pm iN) \ \Big] \ . \qquad (2.4c)
$$

In the same manner for $V^{(J+1/2)}_{\epsilon JM}$ we get

$$
V^{(J+1/2)}_{\epsilon JM\Delta\lambda} = \left| \begin{array}{rrrr} \lambda\,\Omega & \lambda\,\Xi & \Omega & \Xi \\ \lambda\,\Upsilon & \lambda\,Z & \Upsilon & Z \\ \Omega & \Xi & \lambda\,\Omega & \lambda\,\Xi \\ \Upsilon & Z & \lambda\,\Upsilon & \lambda\,Z \end{array} \right| \begin{array}{l} \leftarrow \ H^{+} \\ \leftarrow \ H^{-} \\ \leftarrow \Delta\ H^{+} \\ \leftarrow \Delta\ H^{-} \end{array} , \qquad (2.5a)
$$

where now symbols Ω, Ξ, Υ , Z note the following expressions (compare with (2.4b)):

$$
\Omega = \sqrt{\frac{J-M+1}{J(J+1)}}\, D^{J+1/2}_{-M+1/2,-1/2} \ , \ \ \Xi = \sqrt{\frac{J+M+1}{J(J+1)}}\, D^{J+1/2}_{-M-1/2,-1/2} \ ,
$$

$$
\Upsilon = \sqrt{\frac{J-M+1}{J(J+1)}}\, D^{J+1/2}_{-M+1/2,+1/2} \ , \ \ Z = \sqrt{\frac{J+M+1}{J(J+1)}}\, D^{J+1/2}_{-M-1/2,-1/2} \ ,
$$

(2.5b)

and

$$H^{\pm}(r) = \sqrt{\frac{J}{2J+1}} \left[\sqrt{J+1}\,(K \pm iL) \; + \; \sqrt{J+1}\,(M \pm iN) \right] . \qquad (2.5c)$$

Now we should take into account substitution $(1.8a, b)$, which results in

$$U^{I}_{\epsilon JM\Delta\lambda} \;\rightarrow\; \{\, V^{(J+1/2)}_{\epsilon JM\Delta\lambda} \;,\;\; V^{(J-1/2)}_{\epsilon JM\Delta\lambda} = 0 \,\} \;,$$

$$U^{II}_{\epsilon JM\Delta\lambda} \;\rightarrow\; \{\, V^{(J+1/2)}_{\epsilon JM\Delta\lambda} = 0 \;,\;\; V^{(J-1/2)}_{\epsilon JM\Delta\lambda} \,\} \;. \qquad (2.6)$$

Let us expand rows (4×4)-matrices $V^{(J\pm 1/2)}_{\epsilon JM\Delta\lambda}(x)$ in terms of Dirac solutions $\Psi_{\epsilon jm\delta}(x)$. First consider $V^{(J+1/2)}(x)$. We should take $(\Delta,\ \lambda)$ subsequently as

$$+1, +1; \;\; +1, -1; \;\; -1, +1; \;\; -1, -1$$

and should note in expressions $(2.4a)$ and $(2.4b)$ only signs at $\Omega,\ \Xi,\ Y,\ Z$. Thus, we get

$$V^{(J+1/2)}_{JM,+1,+1} = \left| \begin{array}{cccc} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ + & + & + & + \end{array} \right| \;,\;\; V^{(J+1/2)}_{JM,+1,-1} = \left| \begin{array}{cccc} - & - & + & + \\ - & - & + & + \\ + & + & - & - \\ + & + & - & - \end{array} \right| \;,$$

$$V^{J+1/2}_{JM,-1,+1} = \left| \begin{array}{cccc} + & + & + & + \\ + & + & + & + \\ - & - & - & - \\ - & - & - & - \end{array} \right| \;,\;\; V^{J+1/2}_{JM,-1,-1} = \left| \begin{array}{cccc} - & - & + & + \\ - & - & + & + \\ - & - & + & + \\ - & - & + & + \end{array} \right| \;. \qquad (2.7)$$

The functions f and g entering the matrix $V^{(J+1/2)}_{\epsilon JM,+1,+1}(x)$ obey eqs. $(1.9a)$ (see also $(1.10b)$). Comparing equations for f and g with those for F and G in (2.1), and also noting relevant Wigner functions, we conclude that the row from the matrix (2.7) satisfy the ordinary Dirac equation; at this we find corresponding quantum number explicit form of linear expansions

$$V^{(J+1/2)}_{JM,+1,+1} = \Big\{ \sqrt{\frac{J-M-1}{J(J+1)}} \; \Big[\Psi^{(1)}_{J+1/2,M-1/2,+1} \; + \; \Psi^{(3)}_{J+1/2,M-1/2,+1} \Big]$$

$$+ \sqrt{\frac{J+M+1}{J(J+1)}} \; \Big[\Psi^{(2)}_{J+1/2,M+1/2,+1} \; + \; \Psi^{(4)}_{J+1/2,M+1/2,+1} \Big] \Big\} \;. \qquad (2.8)$$

In the same manner we consider three remaining cases.

Let us introduce the notation

$$j = (J+1/2) \;,\; m = (M+1/2) \;\;,\;\; m' = M - 1/2 \;,$$

$$\alpha = \sqrt{\frac{J-M+1}{J(J+1)}} \;\;,\;\; \beta = \sqrt{\frac{J+M+1}{J(J+1)}} \;\;, \qquad (2.9a)$$

them expansions of the Dirac-Kähler boson solutions in terns of fermion Dirac solutions can be presented as follows

$$V^{(J+1/2)}_{JM,+1,+1} = \alpha\,(\Psi^{(1)}_{jm',+1} \;+\; \Psi^{(3)}_{jm',+1}) \;+\; \beta\,(\Psi^{(2)}_{jm,+1} \;+\; \Psi^{(4)}_{jm,+1}) \;\; ,$$

$$V^{(J+1/2)}_{JM,-1,-1} = \;\alpha\,(-\Psi^{(1)}_{jm',+1} \;+\; \Psi^{(3)}_{jm',+1}) \;+\; \beta\,(-\Psi^{(2)}_{jm,+1} \;+\; \Psi^{(4)}_{jm,+1}) \;\; ,$$

$$V^{(J+1/2)}_{JM,+1,-1} = \;\alpha\,(-\Psi^{(1)}_{jm',-1} \;+\; \Psi^{(3)}_{jm',-1}) \;+\; \beta\,(-\Psi^{(2)}_{jm,-1} \;+\; \Psi^{(4)}_{jm,-1}) \;\; ,$$

$$V^{(J+1/2)}_{JM,-1,+1} = \;\alpha\,(\Psi^{(1)}_{jm',-1} \;+\; \Psi^{(3)}_{jm',-1}) \;+\; \beta\,(\Psi^{(2)}_{jm,-1} \;+\; \Psi^{(4)}_{jm,-1}) \;\; . \qquad (2.9b)$$

Not, let us consider solutions $V^{(J-1/2)}_{JM\Delta\lambda}$ with the structure (only the first row is written down)

$$V^{(J-1/2}_{JM,+1,+1} = \left| \begin{array}{cccc} (f+ig)\,\Omega & . & . & . \\ -(f-ig)\,\Upsilon & . & . & . \\ (f-ig)\,\Omega & . & . & . \\ -(f+ig)\,\Upsilon & . & . & . \end{array} \right| \; ; \qquad (2.10a)$$

where f and g obey

$$(\frac{d}{dr} - \frac{J}{r})f + (\epsilon + m)g = 0 \;\; ,$$

$$(\frac{d}{dr} + \frac{J}{r})g - (\epsilon - m)f = 0 \; . \qquad (2.10b)$$

Note that eqs. $(2.10b)$ do not coincide with relevant equation in Dirac case at $\delta = \pm 1$; besides, in the rows of the matrix $V^{(J-1/2)}_{JM,+1,+1}$ (in $(2.10b)$ on;y the first row is written down) we do not have the structure required to relate them with Dirac solutions

$$\Psi_{jm,\delta=+1} \sim \left| \begin{array}{c} + \\ + \\ + \\ + \end{array} \right| \; , \quad \Psi_{jm,\delta=-1} \sim \left| \begin{array}{c} + \\ + \\ - \\ - \end{array} \right| \; .$$

However, both impediments can be removed by simple change in notation $f = -G, g = +F$. Then

$$(f+ig) = i(F+iG) \qquad (f-ig) = -i(F-iG) \; ,$$

and eqs. $(2.10b)$ read as

$$(\frac{d}{dr} + \frac{J}{r})F + (\epsilon - m)G = 0 \;\; ,$$

$$(\frac{d}{dr} - \frac{J}{r})G - (\epsilon + m)F = 0 \;\; ; \qquad (2.11a)$$

and $(2.10a)$ reduces to

$$V^{(J-1/2)}_{JM,+1,+1} = \left| \begin{array}{cccc} i(F+iG)\,\Omega & . & . & . \\ i(F-iG) ...\Upsilon & . & . & . \\ -i(F-iG)\,\Omega & . & . & . \\ -i(F+iG)\,\Upsilon & . & . & . \end{array} \right| \; . \qquad (2.11b)$$

Further, allowing for (2.11a,b) and structure of the matrix in four cases

$$V^{(J-1/2)}_{JM,+1,+1} = \begin{vmatrix} + & - & + & - \\ + & - & + & - \\ - & + & - & + \\ - & + & - & + \end{vmatrix} , \quad V^{(J-1/2}_{JM,+1,-1} = \begin{vmatrix} - & + & + & - \\ - & + & + & - \\ - & + & + & - \\ - & + & + & - \end{vmatrix} ,$$

$$V^{(J-1/2}_{JM,-1,+1} = \begin{vmatrix} + & - & + & - \\ + & - & + & - \\ + & - & + & - \\ + & - & + & - \end{vmatrix} , \quad V^{(J-1/2)}_{JM,-1,-1} = \begin{vmatrix} - & + & + & - \\ - & + & + & - \\ + & - & - & + \\ + & - & - & + \end{vmatrix} ,$$

(2.12a)

we arrive at needed expansions for $V^{(J-1/2)}_{JM\Delta\lambda}$

$$V^{(J-1/2)}_{JM,+1,-1} = \rho \, (i\Psi^{(1)}_{jm',-1} + i\Psi^{(3)}_{jm',-1}) - \sigma \, (i\Psi^{(2)}_{jm,-1} + i\Psi^{(4)}_{jm,-1}) \, ,$$

$$V^{(J-1/2)}_{JM,-1,-1} = \rho \, (-i\Psi^{(1)}_{jm',-1} + i\Psi^{(3)}_{jm',-1}) + \sigma \, (i\Psi^{(2)}_{jm,-1} - i\Psi^{(4)}_{jm,-1}) \, ,$$

$$V^{(J-1/2}_{JM,+1,-1} = \rho \, (-i\Psi^{(1)}_{jm',+1} + i\Psi^{(3)}_{jm',+1}) + \sigma \, (i\Psi^{(2)}_{jm,+1} - i\Psi^{(4)}_{jm,+1}) \, ,$$

$$V^{(J-1/2)}_{JM,-1,+1} = \rho \, (i\Psi^{(1)}_{jm',+1} + i\Psi^{(3)}_{jm',+1}) - \sigma \, (i\Psi^{(2)}_{jm,+1} + i\Psi^{(4)}_{jm,+1}) \, ,$$

(2.12b)

where

$$j = (J-1/2) \;\; , \;\; \rho = \sqrt{\frac{J+M}{J(J+1)}} \;\; , \;\; \sigma = \sqrt{\frac{J-M}{J(J+1)}} \, .$$

Expansions for the case of minimal value $J = 0$ will be much more simple

$$V_{00,+1,+1} = \Psi^{(1)}_{1/2,-1/2,+1} + \Psi^{(3)}_{1/2,-1/2,+1} + \Psi^{(2)}_{1/2,+1/2,+1} + \Psi^{(4)}_{1/2,+1/2,+1} \, ,$$

$$V_{00,-1,-1} = -\Psi^{(1)}_{1/2,-1/2,+1} + \Psi^{(3)}_{1/2,-1/2,+1} + -\Psi^{(2)}_{1/2,+1/2,+1} + \Psi^{(4)}_{1/2,+1/2,+1} \, ,$$

$$V_{00,+1,+1} = -\Psi^{(1)}_{1/2,-1/2,-1} + \Psi^{(3)}_{1/2,-1/2,-1} + -\Psi^{(2)}_{1/2,+1/2,-1} + \Psi^{(4)}_{1/2,+1/2,-1} \, ,$$

$$V_{00,+1,+1} = \Psi^{(1)}_{1/2,-1/2,-1} + \Psi^{(3)}_{1/2,-1/2,-1} + \Psi^{(2)}_{1/2,+1/2,-1} + \Psi^{(4)}_{1/2,+1/2,-1} \, . \qquad (2.13)$$

3 Discussion

Above, after performing a special transformation over 4×4-matrix $U(x) \Longrightarrow V(x)$, spherical boson solution of the Dirac–Kähler equation, simple linear expansions of the four rows of new representative of the Dirac–Kähler field $V(x)$ in terms of spherical fermion solutions $\Psi_i(x)$ of the four ordinary Dirac equation have been derived. However, this fact cannot be interpreted as the possibility not to distinguish between the Dirac-Kähler fileld and the system four Dirac fermions. The main formal argument is that the special transformation $(S(x) \otimes I)$ involved does not belong to the group of tetrad local gauge transformation for Dirac-Kähler field, 2-rank

bispinor under the Lorentz group. Therefore, the linear expansions between boson and fermion functions are not gauge invariant under the group of local tetrad rotations.

Formal possibility to produce such expansions exists only for the case of flat Minkowski space-time, and cannot be extended to any other space-time with curvature. For instance, let us specify the situation for spherical space with constant positive curvature In spherical coordinates and tetrad

$$dS^2 = dt^2 - d\chi^2 - \sin^2\chi(\, d\theta^2 + \sin^2\theta d\phi^2 \,) \; ,$$

$$e^{\alpha}_{(0)} = (1, \; 0, \; 0, \; 0) \; , \;\; e^{\alpha}_{(1)} = (0, \; 0, \; \sin^{-1}\chi, \; 0) \; ,$$

$$e^{\alpha}_{(2)} = (0, \; 0, \; 0, \; \sin^{-1}\chi \sin^{-1}\theta) \; , \; e^{\alpha}_{(3)} = (0,1,0,0) \; , \qquad (3.1)$$

the Dirac–Kähler equation takes the form

$$\left[\, i\gamma^0 \, \partial_t \; + \; i \left(\gamma^3 \, \partial_\chi \; + \; \frac{\gamma^1 J^{31} + \gamma^2 J^{32}}{\mathrm{tg} \; \chi} \right) \; + \; \frac{1}{\sin\chi} \, \Sigma_{\theta,\phi} - m \right] \, U(x) = 0 \; . \qquad (3.2)$$

Most of calculations performed above are valid here with small formal changes; in particular, instead of (1.7b) we have at $\Delta = +1 \; , \;\; \lambda = +1$

$$\frac{dK}{d\chi} + \frac{a}{\sin\chi} M + (\epsilon + m) L = 0 \; ,$$

$$\frac{dL}{d\chi} - \frac{a}{\sin\chi} N - (\epsilon - m) K = 0 \; ,$$

$$(\frac{d}{d\chi} + \frac{1}{\mathrm{tg} \; \chi}) M + \frac{a}{\sin\chi} K + (\epsilon + m) N = 0 \; ,$$

$$(\frac{d}{d\chi} - \frac{1}{\mathrm{tg} \; \chi}) N - \frac{a}{\sin\chi} L - (\epsilon - m) M = 0 \; . \qquad (3.3)$$

However, the above used substitutions of the form (1.8a, b), cannon be imposed because they are not consistent with eqs. (3.3. This means that no solutions for radial functions of (formally) fermion type exist in spherical space. The latter is due to the fact that one cannot find any transformation like $I \otimes S(x)$ which would divide the Dirac–Kähler equation into four separate Dirac equations.

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